MATHEMATICS METHODS

MAWA Semester 2 (Unit 3&4) Examination 2018
Calculator-assumed

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, Fri October 12th 2018**

Section Two: Calculator-assumed (100 Marks)

Question 8 (a) (2 marks)

Solution	
Bernoulli distribution with parameter $p = \frac{1}{300}$	
Mathematical behaviours	**Marks**
• states Bernoulli distribution	1
• states parameter of $p = \frac{1}{300}$	1

Question 8 (b) (2 marks)

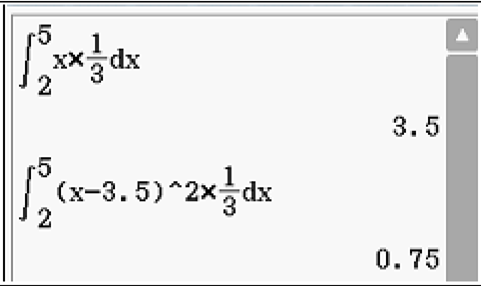
Solution	
$E(X) = \frac{1}{300}$ Variance(X) $= \frac{1}{300}\left(1 - \frac{1}{300}\right)$ $= \frac{1}{300}\left(\frac{299}{300}\right)$ $= \frac{299}{90000}$	
Mathematical behaviours	**Marks**
• states correct mean	1
• states variance	1

Question 9 (a) **(2 marks)**

Solution

$$E(X)=\mu=\int_{-\infty}^{\infty} x\times p(x)\,dx$$
$$=\int_{2}^{5} x\times\frac{1}{3}\,dx$$
$$=3.5$$

$$\sigma^2=\int_{2}^{5}(x-3.5)^2\times\frac{1}{3}\,dx$$
$$=0.75$$



Mathematical behaviours	Marks
• determines the expected value	1
• determines the variance	1

Question 9 (b) **(2 marks)**

Solution

$$P(x\le 4\mid x>3)=\frac{P(3<x\le 4)}{P(x>3)}$$
$$=\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{1}{2}$$

Mathematical behaviours	Marks
• determines numerator	1
• determines denominator	1

Question 9 (c) **(3 marks)**

Solution

$$F(x)=\begin{cases}0 & x<2\\ \displaystyle\int_{2}^{x}\frac{1}{3}\,dx=\frac{1}{3}(x-2) & 2\le x\le 5\\ 1 & x>5\end{cases}$$

Mathematical behaviours	Marks
• identifies need to integrate $f(x)$	1
• determines definite integral using correct limits of integration $(2,x)$	1
• determines $F(x)$ and states it as a piecewise function	1

Question 9 (d) **(2 marks)**

Solution

$$F(x)=0.75 \Rightarrow 0.75=\frac{1}{3}(x-2)$$

ie $x=4.25$

Hence the upper quartile is 4.25.

Mathematical behaviours	Marks
• states $0.75=\frac{1}{3}(x-2)$	1
• solves for x	1

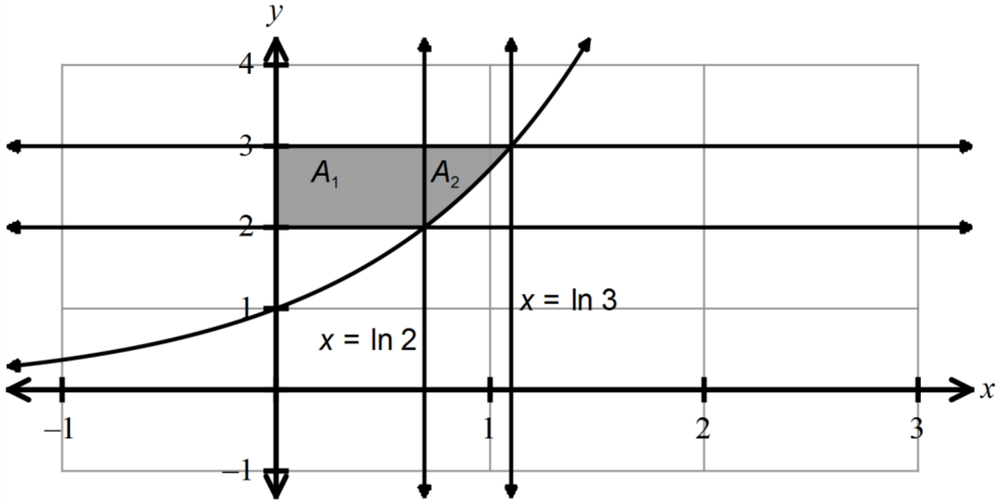
Question 10 (a) **(2 marks)**

Solution	
$P = P_o e^{kt}$ ie $P = 35e^{0.03t}$ $P = 50 \Rightarrow 50 = 35e^{0.03t}$ ie $t = 11.88$ ie the population will reach 50 million in roughly 11 years and 11 months	
Mathematical behaviours	**Marks**
• substitutes $P_0 = 35$ and $P = 50$ to obtain required equation • solves equation to obtain t	1 1

Question 10 (b) **(2 marks)**

Solution	
In 15 years the country's population = $35e^{15\times 0.03}$ In 15 years the city's population = $(0.22)35e^{c\times 15}$ where c represents its growth rate Hence $(0.22)35e^{c\times 15} = (0.4)35e^{15\times 0.03}$ Solving gives $c \approx 0.0699$ Hence the continuous growth rate is approximately 7%.	
Mathematical behaviours	**Marks**
• equates city's population to 40% of country's population in 15 years • solves equation and states percentage growth rate	1 1

Question 11 **(6 marks)**

Solution

Total area $= A_1 + A_2$ $= (\ln 2)(3-2) + 3(\ln 3 - \ln 2) - \int_{\ln 2}^{\ln 3} e^x dx$ $= \ln 2 + 3\ln 3 - 3\ln 2 - \left[e^x\right]_{\ln 2}^{\ln 3}$ $= 3\ln 3 - 2\ln 2 - (3-2)$ $= \ln 3^3 - \ln 2^2 - 1$ $= \ln \frac{27}{4} - 1$

Mathematical behaviours	Marks
• indicates an appropriate expression involving an integral to determine required area	1
• indicates x axis values, ln 2 and ln 3	1
• determines A_1	1
• substitutes correct bounds to determine A_2	1
• evaluates integral and A_2	1
• rearranges expression using log laws and simplifies	1

Question 12 (a) **(1 mark)**

Solution	
Confidence interval is $(\hat{p}-E,\hat{p}+E)=(0.53,0.61)$ So $\hat{p}=\frac{0.53+0.61}{2}=0.57$	
Mathematical behaviours	Marks
• Obtains correct answer	1

Question 12 (b) **(2 marks)**

Solution	
$E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ ie $0.04=1.96\sqrt{\frac{0.57\times0.43}{n}}$ Solving for n gives $n\cong588.5$ So the sample size was 589 (approximately)	
Mathematical behaviours	Marks
• Uses $Z_\alpha\approx1.96$ • Solves for n and rounds	1 1

Question 12 (c) **(3 marks)**

Solution	
$E=z_\alpha\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $0.07=z_\alpha\sqrt{\frac{0.57(0.43)}{589}}$ $z_\alpha=3.43$ $P(-3.43<Z<3.43)=0.9993964$ Hence the level of confidence is 99.94%	
Mathematical behaviours	Marks
• Substitutesvalues into error equation • Solves for z_α • States level of confidence to at least 1 decimal place	1 1 1

Question 12 (d) **(2 marks)**

Solution	
From the confidence interval in 10(c), there is a 99.94% probability that p lies between 0.5 and 0.64, and in particular $p>0.5$. So the claim is justified.	
Mathematical behaviours	Marks
• States the claim is justified • Gives a valid reason	1 1

Question 13 (a) **(3 marks)**

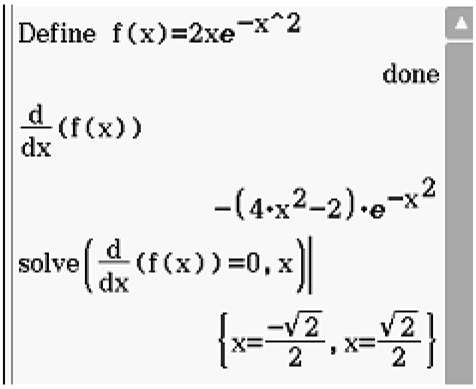
Solution	
$v(t)=3t^2+2t-9$ $v(0)=9$ $a(t)=6t-2$ $a(0)=-2$ Since velocity and acceleration are opposing one another, the particle is slowing down	
Mathematical behaviours	Marks
• calculates $v(0)$ • differentiates $v(t)$ to obtain $a(t)$ and $a(0)$ • states particle is slowing down	1 1 1

Question 13 (b) **(2 marks)**

Solution	
$v(t)=3t^2-2t+9,$ $\therefore s(t)=t^3-t^2+9t+c$ $(0,-1)\Rightarrow c=-1$ $\therefore s(t)=t^3-t^2+9t-1$ $s(5)=5^3-5^2+9(5)-1=144$ Hence its final position is 144m from the origin.	
Mathematical behaviours	Marks
• integrates velocity equation to determine displacement equation including $c=-1$ • substitutes $t=5$, calculates and states final position, with unit	1 1

Question 14 **(4 marks)**

Solution
$A(x)=2x.e^{-x^2}$ $\dfrac{dA}{dx}=2\left(-2x^2e^{-x^2}+e^{-x^2}\right)=2e^{-x^2}(-2x^2+1)$ $\dfrac{dA}{dx}=0\Rightarrow x=\pm\dfrac{1}{\sqrt{2}}$ $x=\dfrac{1}{\sqrt{2}}\Rightarrow y=e^{-\frac{1}{2}}$ Hence C has coordinates $\left(\dfrac{1}{\sqrt{2}},e^{-\frac{1}{2}}\right)$



Mathematical behaviours	Marks
• states $A(x)$	1
• differentiates $A(x)$	1
• solves $\dfrac{dA}{dx}=0$	1
• states co-ordinates of C	1

Question 15 (a) **(1 mark)**

Solution	
Probability of winning a prize = 0.1 + 0.001 = 0.101	
Mathematical behaviours	**Marks**
• uses Addition Principle to calculate the correct probability	1

Question 15 (b) **(2 marks)**

Solution	
$X \sim Bin(20, 0.101)$	
Mathematical behaviours	**Marks**
• states Binomial distribution	1
• states correct parameters	1

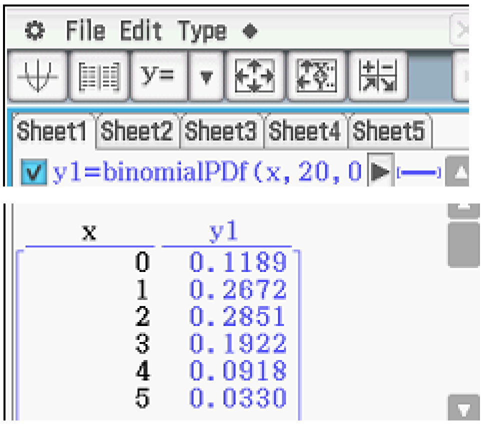
Question 15 (c) **(2 marks)**

Solution	
$P(X \le 3) = \binom{20}{0}(0.101)^0(0.899)^{20} + \binom{20}{1}(0.101)^1(0.899)^{19} + \binom{20}{2}(0.101)^2(0.899)^{18} + \binom{20}{3}(0.101)^3(0.899)^{17}$	
Mathematical behaviours	**Marks**
• identifies that “no more than 3 prizes” means “can win 0,1,2, or 3 prizes”	1
• states correct expression for the probability	1

Question 15 (d) **(1 mark)**

Solution	
$P(X \le 3)$ Bincdf(0,3,20,0.101)=0.8634	
Mathematical behaviours	**Marks**
• states probability	1

Question 15 (e) **(1 mark)**

Solution	
Using CAS, k=3 	
Mathematical behaviours	**Marks**
• states k=3	1

Question 16 (a) (i) **(2 marks)**

Solution	
$\int_{-3}^{2} f(x)\, dx = -3 + 4 - 4 = -3$	
Mathematical behaviours	Marks
• indicates addition of signed areas • determines result	1 1

Question 16 (a) (ii) **(2 marks)**

Solution	
Area = 3+4+4=11	
Mathematical behaviours	Marks
• expresses the area as a sum of areas • determines result	1 1

Question 16 (b) **(3 marks)**

Solution	
$\int_{2}^{0}\left(x-2f(x)\right)dx = \int_{2}^{0} x\ dx + 2\int_{0}^{2} f(x)\ dx$ $=\left[\frac{x^2}{2}\right]_2^0 + 2(-4)$ $=\frac{1}{2}(0-4)-8=-10$	
Mathematical behaviours	Marks
• applies the additivity of integrals to split the integral • applies the linearity of integrals to deduce $\int_{2}^{0}\left(-2f(x)\right)dx = 2\int_{0}^{2} f(x)\ dx$ • determines result	1 1 1

Question 16 (c) **(3 marks)**

Solution	
Maximum value of $F(x)$ occurs where $F'(x)=0$ $F'(x)=\frac{d}{dx}\int_{-3}^{x} f(t)\ dt = f(x)$ $f(x)=0 \Rightarrow x=-2,0,2$ $F(-2)=-3, F(0)=1, F(2)=-3$ Hence, max is 1.	
Mathematical behaviours	Marks
• applies the Fundamental Theorem • solves $f(x)=0,$ stating $x=-2,0,2$ • determines maximum value	1 1 1

Question 17 (a) **(1 mark)**

Solution
$D \sim N(65, 4.9^2)$

	small	medium	large
Proportion of peaches	$P(d < 57) = 0.0513$	$P(57 \le d \le 70) = 0.7950$	0.1538

Mathematical behaviours	Marks
• determines both probabilities	1

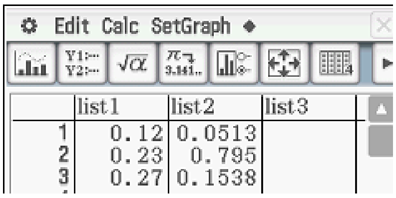
Question 17 (b) **(3 marks)**

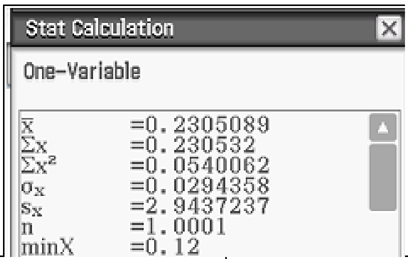
Solution

Let X = profit

$$\mu = \sum xp(x)$$
$$= 0.12 \times 0.0513 + 0.23 \times 0.795 + 0.27 \times 0.1538$$
$$= 0.2305$$
$$\approx 23c$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$
$$= (0.12-0.23)^2 \times 0.0513 + (0.23-0.23)^2 \times 0.795 + (0.27-0.23)^2 \times 0.1538$$
$$= 0.00087$$
$$\sigma = 0.0294$$
$$\approx 2.9c$$





Mathematical behaviours	Marks
• states a calculation to determine the mean or variance	1
• determines mean	1
• determines standard deviation	1

Question 17 (c) **(3 marks)**

Solution

Let $T = 25X - 0.15$

$E(T) = 25E(X) - 0.15 = 25(0.2305) - 0.15 = 5.6125 \approx \$5.61$

$Std\ Dev(T) = 25 \times Std\ Dev(X) = 25 \times 0.0294 = 0.735 \approx 74c$

Mathematical behaviours	Marks
• states linear transformation required	1
• determines mean	1
• determines standard deviation	1

Question 18 (a) **(4 marks)**

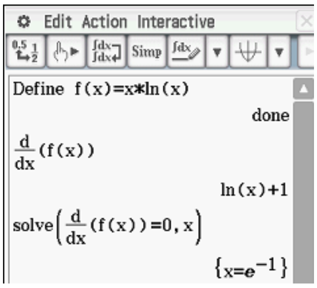
Solution

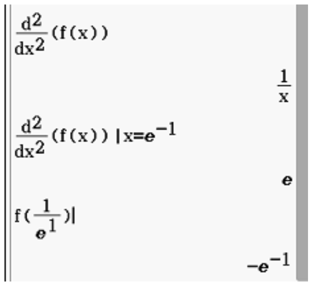
$y = x\ln x$

$\dfrac{dy}{dx} = (\ln x)(1) + x.\dfrac{1}{x}$

$\dfrac{dy}{dx} = \ln x + 1$

$\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$





For stationary points,

$\ln x + 1 = 0$

ie $\ln x = -1$

ie $x = e^{-1}$

$x = e^{-1} \Rightarrow y = -\dfrac{1}{e}$

$\dfrac{d^2y}{dx^2} = \dfrac{1}{e^{-1}} = e > 0,$ hence a minimum turning point

Thus exact coordinates of minimum turning point = $\left(\dfrac{1}{e}, \dfrac{-1}{e}\right)$

Mathematical behaviours	Marks
• differentiates correctly using the product rule	1
• equates first derivative to zero to determine x co-ordinate of stationary points	1
• uses second derivative test to determine nature of turning point	1
• determines correct coordinates of turning point	1

Question 18 (b) **(1 mark)**

Solution

Since

$\dfrac{d^2y}{dx^2} = \dfrac{1}{x} \neq 0$ for any x value.

Hence the curve cannot have a point of inflection.

Mathematical behaviours	Marks
• states second derivative is never zero hence no P.O.I.	1

Question 19 (a) **(2 marks)**

Solution	
$dB = 10\log_{10}(\frac{I}{I_0})$ $60 = 10\log_{10}(\frac{I}{1\times10^{-12}})$ $I = 10^{-6}$ Watts/m^2	
Mathematical behaviours	Marks
• substitutes $dB = 60$ • solves the equation	1 1

Question 19 (b) **(2 marks)**

Solution	
$k = 10\log_{10}(\frac{I}{I_0}) \Rightarrow 0.1k = \log_{10}(\frac{I}{I_0}) \Rightarrow 10^{(0.1k)} = \frac{I}{I_0} \Rightarrow I = 10^{(0.1k)} I_0$ $k = 85 \Rightarrow I_g = 10^{(0.1\times85)} I_0 \Rightarrow I_g = 10^{8.5} I_0$	
Mathematical behaviours	Marks
• rearranges log expression correctly • substitutes $k = 85$ and states I_g in terms of I_o	1 1

Question 19 (c) **(1 mark)**

Solution	
$\frac{I_g}{I_c} = \frac{10^{8.5} I_0}{10^6 I_0} = 10^{2.5} \approx 316$ ie the gunshot is approximately 316 times louder than the restaurant conversation	
Mathematical behaviours	Marks
• evaluates the ratio $\frac{I_g}{I_c}$ correctly	1

Question 20 (a) **(6 marks)**

Solution

$y(9)=0 \Rightarrow ae^{-bt}\sin 9c = 0 \Rightarrow 9c=\pi$

$\Rightarrow c=\dfrac{\pi}{9}\approx 0.349$

$\dfrac{dy}{dt} = -abe^{-bt}\sin ct + ace^{-bt}\cos ct = ae^{-bt}(c\cos ct - b\sin ct)$

max at $t=4$, $\Rightarrow \dfrac{dy}{dt}\Big|_{t=4}=0$

ie $-abe^{-bt}\sin ct + ace^{-bt}\cos ct = 0$

ie $-ae^{-bt}(b\sin ct - c\cos ct)=0$

ie $b\sin ct - c\cos ct = 0$

ie $b\sin(4\times 0.349) - 0.439\cos(4\times 0.349)=0 \Rightarrow b=0.0616$

$y(4)=60 \Rightarrow ae^{-(0.0616\times 4)}\sin(4\times 0.349)=60 \Rightarrow a=78.0.$

```
Define f(t)=ae^(-bt)sin(ct)
                                        done
solve(f(9)=0,c)
  {c=0.3490658504·constn(1)}
```

```
d/dt(f(t))
-(a·b·sin(c·t)-a·c·cos(c·t))·e
solve(d/dt(f(t))=0|t=4, b)
                     {b=c·cos(4·c)/sin(4·c)}
0.349·cos(4·0.349)/sin(4·0.349)
                           0.06163290595
```

```
Define f(t)=ae^(-0.0616t)sin(0
                                        done
solve(f(4)=60,a)
                          {a=77.95251617}
```

Mathematical behaviours	Marks
• uses (9,0) to conclude $9c=\pi$	1
• solves for c	1
• determines derivative function	1
• equates derivative function to 0 at $t=4$ and solves for b	1
• uses (4,60) to solve for a	1
• gives all answers correct to 3 significant figures	1

Question 20 (b) **(3 marks)**

Solution

$y\left(t+\frac{\pi}{c}\right) = ae^{-b\left(t+\frac{\pi}{c}\right)}\sin\left(c\left(t+\frac{\pi}{c}\right)\right)$

$= ae^{-bt}e^{-\frac{b\pi}{c}}\sin(ct+\pi)$

$= -ae^{-bt}e^{-\frac{b\pi}{c}}\sin ct = -ry(t)$

Mathematical behaviours	Marks
• replaces t with $t+\dfrac{\pi}{c}$ to obtain required function	1
• uses indices laws to factor out ae^{-bt}	1
• uses $\sin(ct+\pi)=-\sin ct$ to complete argument	1

Question 20 (c) **(3 marks)**

Solution	
The mass travels 120 cm in the first 9 seconds. In the next 9-second period the mass travels $120r$ cm, where from part (b) $r = e^{-\pi b/c} \cong e^{-\pi \times \frac{0.0616}{0.349}} \cong 0.574$ Hence distance travelled between its first and second return to the origin is $0.574 \times 120 = 68.88\,\text{cm} \approx 69\text{cm}$	
Mathematical behaviours	**Marks**
• deduces that the mass travels 120 m in the first 9 seconds • evaluates r • obtains correct answer to the nearest centimetre	1 1 1

Question 21 (a) **(3 marks)**

Solution	
$\hat{p} = \frac{465}{700} \approx 0.6643, \quad E = 1.96\sqrt{\frac{(0.6643)(1-0.6643)}{700}} \approx 0.0350$ CI is $0.6643 - 0.0350 < p < 0.6643 + 0.0350$ ie $0.6293 < p < 0.6993$ So the 95% confidence interval is $0.629 < p < 0.699$	
Mathematical behaviours	**Marks**
• calculates sample proportion correctly • calculate standard error correctly • calculates interval correctly	1 1 1

Question 21 (b) **(2 marks)**

Solution	
Because the old satisfaction rate (65%) lies within the new confidence interval, the recent survey does not provide conclusive evidence that the satisfaction rate has improved.	
Mathematical behaviours	**Marks**
• states that survey is not conclusive • states a valid reason	1 1

Question 21 (c) (i) **(3 marks)**

Solution	
Since $\hat{p}(1-\hat{p}) \leq \frac{1}{4}$ $E_{95\%} = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 1.96\sqrt{\frac{1}{4n}} = \frac{1.96}{2}\left(\frac{1}{\sqrt{n}}\right) < \frac{1}{\sqrt{n}}$	
Mathematical behaviours	**Marks**
• substitutes $\hat{p}(1-\hat{p}) = 0.25$ • expresses error as a constant $\times \frac{1}{\sqrt{n}}$ • deduces error $< \frac{1}{\sqrt{n}}$	1 1 1

Question 21 (c) (ii) **(1 mark)**

Solution	
$E = \frac{1}{\sqrt{n}}$	
Mathematical behaviours	**Marks**
• states Margin of error	1

Question 21 (c) (iii) **(1 mark)**

Solution	
$E_{95\%} < E_{\text{simple interval}}$ Hence the interval $\hat{p} - E < p < \hat{p} + E$ is a subset of the interval $\hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$ and hence the confidence interval of $\hat{p} - \frac{1}{\sqrt{n}} < p < \hat{p} + \frac{1}{\sqrt{n}}$ is at least 95%	
Mathematical behaviours	**Marks**
• uses the relationship between the errors to draw valid conclusion	1

Question 21 (d) **(3 marks)**

Solution	
True margin of error $E = z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ If $E = \frac{1}{\sqrt{n}}$ and $\hat{p} = 0.7$, $\frac{1}{\sqrt{n}} = z_{\alpha}\sqrt{\frac{(0.7)(0.3)}{n}}$ $ie \quad \frac{1}{\sqrt{n}} = z_{\alpha}\frac{\sqrt{0.21}}{\sqrt{n}} \Rightarrow z_{\alpha} \approx 2.18$ $P(-2.18 < z < 2.18) = 0.9707$ Hence level of confidence is approximately 97%	
Mathematical behaviours	**Marks**
• substitute $\hat{p} = 0.7$ and equates true E to E for simple interval • solves inequality to determine z • states 97% level of confidence	1 1 1

Question 21 (e) **(1 mark)**

Solution	
$\frac{1}{\sqrt{n}} = 0.04 \Rightarrow n = 625$ Hence a sample size of 700 is appropriate	
Mathematical behaviours	**Marks**
• calculates n and deduces that sample size is appropriate	1